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NEW CONGRUENCES INVOLVING HARMONIC NUMBERS

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ABSTRACT. Let $p > 3$ be a prime. For any p -adic integer a , we determine

$$\sum_{k=0}^{p-1} \binom{-a}{k} \binom{a-1}{k} H_k, \quad \sum_{k=0}^{p-1} \binom{-a}{k} \binom{a-1}{k} H_k^{(2)}, \quad \sum_{k=0}^{p-1} \binom{-a}{k} \binom{a-1}{k} \frac{H_k^{(2)}}{2k+1}$$

modulo p^2 , where $H_k = \sum_{0 < j \leq k} 1/j$ and $H_k^{(2)} = \sum_{0 < j \leq k} 1/j^2$. In particular, we show that

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{-a}{k} \binom{a-1}{k} H_k &\equiv (-1)^{\langle a \rangle_p} 2 (B_{p-1}(a) - B_{p-1}) \pmod{p}, \\ \sum_{k=0}^{p-1} \binom{-a}{k} \binom{a-1}{k} H_k^{(2)} &\equiv -E_{p-3}(a) \pmod{p}, \\ (2a-1) \sum_{k=0}^{p-1} \binom{-a}{k} \binom{a-1}{k} \frac{H_k^{(2)}}{2k+1} &\equiv B_{p-2}(a) \pmod{p}, \end{aligned}$$

where $\langle a \rangle_p$ stands for the least nonnegative integer r with $a \equiv r \pmod{p}$, and $B_n(x)$ and $E_n(x)$ denote the Bernoulli polynomial of degree n and the Euler polynomial of degree n respectively.

1. INTRODUCTION

A classical theorem of J. Wolstenholme [W] asserts that for any prime $p > 3$ we have

$$\frac{1}{2} \binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 \pmod{p^3},$$

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which follows from the congruences

$$H_{p-1} \equiv 0 \pmod{p^2} \quad \text{and} \quad H_{p-1}^{(2)} \equiv 0 \pmod{p},$$

where

$$H_n := \sum_{0 < k \leq n} \frac{1}{k} \quad \text{and} \quad H_n^{(2)} := \sum_{0 < k \leq n} \frac{1}{k^2} \quad \text{for } n \in \mathbb{N} = \{0, 1, 2, \dots\}.$$

Those H_n ($n \in \mathbb{N}$) are the usual harmonic numbers, and those $H_n^{(2)}$ ($n \in \mathbb{N}$) are called second-order harmonic numbers. For recent congruences involving harmonic numbers, one may consult [Su12] and [SZ].

In 2003, based on his analysis of the p -adic analogues of Gaussian hypergeometric series and the Calabi-Yau manifolds, F. Rodriguez-Villegas [RV] conjectured that for any prime $p > 3$ we have

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} &\equiv \left(\frac{-1}{p}\right) \pmod{p^2}, & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} &\equiv \left(\frac{p}{3}\right) \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{64^k} &\equiv \left(\frac{-2}{p}\right) \pmod{p^2}, & \sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} &\equiv \left(\frac{-1}{p}\right) \pmod{p^2}, \end{aligned}$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. All the four congruences were proved by E. Mortenson [M1, M2] via the p -adic Γ -function and modular forms. Recently, Z.-H. Sun [S1] presented elementary proofs of them, and V.J.W. Guo and J. Zeng [GZ] obtained a q -analogue of the first one.

Let $p > 3$ be a prime. The author [Su11] showed that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) - p^2 E_{p-3} \pmod{p^3} \quad (1.1)$$

(see also [Su13] for a simpler proof), and conjectured that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} \equiv \left(\frac{p}{3}\right) - \frac{p^2}{3} B_{p-2} \left(\frac{1}{3}\right) \pmod{p^3}, \quad (1.2)$$

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} \equiv \left(\frac{-2}{p}\right) - \frac{3}{16} p^2 E_{p-3} \left(\frac{1}{4}\right) \pmod{p^3}, \quad (1.3)$$

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p}\right) - \frac{25}{9} p^2 E_{p-3} \pmod{p^3}, \quad (1.4)$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{(2k+1)27^k} \equiv \left(\frac{p}{3}\right) - \frac{2}{3} p^2 B_{p-2} \left(\frac{1}{3}\right) \pmod{p^3}, \quad (1.5)$$

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{(2k+1)64^k} \equiv \left(\frac{-1}{p}\right) - 3p^2 E_{p-3} \pmod{p^3}, \quad (1.6)$$

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{(2k+1)432^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}, \quad (1.7)$$

where E_0, E_1, E_2, \dots are Euler numbers, and $E_n(x)$ denotes the Euler polynomial of degree n given by

$$E_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k},$$

and $B_n(x)$ stands for the Bernoulli polynomial of degree n given by

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k}$$

with B_0, B_1, B_2, \dots Bernoulli numbers. The conjectural congruences (1.2)-(1.7) were recently confirmed by Z.-H. Sun [S2].

In this paper we mainly establish two new theorems involving harmonic numbers and second-order harmonic numbers.

For a prime p and a p -adic integer a , we write $\langle a \rangle_p$ for the unique integer $r \in \{0, 1, \dots, p-1\}$ with $a \equiv r \pmod{p}$, and let $q_p(a)$ denote the Fermat quotient $(a^{p-1} - 1)/p$ if $a \not\equiv 0 \pmod{p}$.

Theorem 1.1. *Let $p > 3$ be a prime. For any p -adic integer a , we have*

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{-a}{k} \binom{a-1}{k} H_k &\equiv (-1)^{\langle a \rangle_p - 1} 2 \sum_{0 < k < \langle a \rangle_p} \frac{1}{a - k} \\ &\equiv (-1)^{\langle a-1 \rangle_p} (2H_{\langle a-1 \rangle_p} + (a - \langle a \rangle_p) B_{p-2}(a)) \pmod{p^2} \\ &\equiv (-1)^{\langle a \rangle_p} 2(B_{p-1}(a) - B_{p-1}) \pmod{p}. \end{aligned} \quad (1.8)$$

Remark 1.1. Let $p > 3$ be a prime and let a be a p -adic integer. Congruences involving the general sum $\sum_{k=0}^{p-1} \binom{a}{k} \binom{a+k}{k} / m^k = \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} / (-m)^k$ with $m \not\equiv 0 \pmod{p}$ first appeared in the author's paper [Su14]. Z.-H. Sun [S1, Corollary 2.1] determined $\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k}$ modulo p^2 with the special cases $a = -1/2, -1/3, -1/4, -1/6$ first discovered by Rodriguez-Villegas [RV]. Besides Theorem 1.1, we are also able to show that

$$\sum_{k=1}^{p-1} \binom{-a}{k} \binom{a-1}{k} \frac{H_k}{k} \equiv (-1)^{\langle -a \rangle_p} E_{p-3}(a) \pmod{p}.$$

Let $p > 3$ be a prime. As

$$\binom{p-1}{k} (-1)^k = \prod_{0 < j \leq k} \left(1 - \frac{p}{j}\right) \equiv 1 - pH_k \pmod{p^2} \quad \text{for all } k = 0, 1, 2, \dots,$$

combining Theorem 1.1 with [S1, Corollary 2.1], we obtain

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{p-1}{k} \binom{-a}{k} \binom{a-1}{k} (-1)^k \\ & \equiv (-1)^{\langle -a \rangle_p} (1 + 2p(B_{p-1}(a) - B_{p-1})) \pmod{p^2} \end{aligned} \tag{1.9}$$

for any p -adic integer a . For each $d = 2, 3, 4, 6$ and any $c \in \{1, \dots, d\}$ with $(c, d) = 1$, E. Lehmer [L] determined $B_{p-1}(c/d) - B_{p-1}$ modulo p in terms of Fermat quotients. For $d \in \{5, 8, 10, 12\}$ and $c \in \{1, \dots, d\}$ with $(c, d) = 1$, A. Granville and the author [GS] determined $B_{p-1}(c/d) - B_{p-1} \pmod{p}$ by showing that

$$\begin{aligned} B_{p-1}\left(\frac{c}{5}\right) - B_{p-1} & \equiv \frac{5}{4} \left(\left(\frac{cp}{5}\right) \frac{1}{p} F_{p-(\frac{5}{p})} + q_p(5) \right) \pmod{p}, \\ B_{p-1}\left(\frac{c}{8}\right) - B_{p-1} & \equiv \left(\frac{2}{cp}\right) \frac{2}{p} P_{p-(\frac{2}{p})} + 4q_p(2) \pmod{p}, \\ B_{p-1}\left(\frac{a}{10}\right) - B_{p-1} & \equiv \frac{15}{4} \left(\frac{cp}{5}\right) \frac{1}{p} F_{p-(\frac{5}{p})} + \frac{5}{4} q_p(5) + 2q_p(2) \pmod{p}, \\ B_{p-1}\left(\frac{c}{12}\right) - B_{p-1} & \equiv \left(\frac{3}{c}\right) \frac{3}{p} S_{p-(\frac{3}{p})} + 3q_p(2) + \frac{3}{2} q_p(3) \pmod{p}, \end{aligned}$$

where $(-)$ is the Jacobi symbol, and the Fibonacci sequence $(F_n)_{n \geq 0}$, the Pell sequence $(P_n)_{n \geq 0}$, and the sequence $(S_n)_{n \geq 0}$ (cf. [Su02]) are defined as follows:

$$\begin{aligned} F_0 &= 0, \quad F_1 = 1, \quad \text{and } F_{n+1} = F_n + F_{n-1} \quad \text{for all } n = 1, 2, 3, \dots; \\ P_0 &= 0, \quad P_1 = 1, \quad \text{and } P_{n+1} = 2P_n + P_{n-1} \quad \text{for all } n = 1, 2, 3, \dots; \\ S_0 &= 0, \quad S_1 = 1, \quad \text{and } S_{n+1} = 4S_n - S_{n-1} \quad \text{for all } n = 1, 2, 3, \dots. \end{aligned}$$

Corollary 1.1. *Let $p > 3$ be a prime. Then*

$$\left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} H_k \equiv -2q_p(2) + p q_p(2)^2 \pmod{p^2}, \quad (1.10)$$

$$\left(\frac{p}{3}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} H_k \equiv -3q_p(3) + \frac{3}{2} p q_p(3)^2 \pmod{p^2} \quad (1.11)$$

$$\left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} H_k \equiv -6q_p(2) + 3p q_p(2)^2 \pmod{p^2}, \quad (1.12)$$

$$\left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} H_k \equiv -3q_p(3) - 4q_p(2) + p \left(\frac{3}{2} q_p(3)^2 + 2q_p(2)^2 \right) \pmod{p^2}. \quad (1.13)$$

Theorem 1.2. *Let $p > 3$ be a prime, and let a be a p -adic integer.*

(i) *If m is a positive integer with $a \equiv m \pmod{p^2}$, then*

$$\sum_{k=0}^{p-1} \binom{-a}{k} \binom{a-1}{k} H_k^{(2)} \equiv 2 \sum_{\substack{0 < k < m \\ p \nmid k}} \frac{(-1)^{m-k}}{k^2} \equiv -E_{p^2-p-2}(a) \pmod{p^2} \quad (1.14)$$

and

$$\begin{aligned} & (2a-1) \sum_{k=0}^{p-1} \binom{-a}{k} \binom{a-1}{k} \frac{H_k^{(2)}}{2k+1} \\ & \equiv -2 \sum_{\substack{0 < k < m \\ p \nmid k}} \frac{1}{k^2} \equiv (2-2p) B_{p^2-p-1}(a) \pmod{p^2}. \end{aligned} \quad (1.15)$$

(ii) *We always have*

$$\sum_{k=0}^{p-1} \binom{-a}{k} \binom{a-1}{k} H_k^{(2)} \equiv -E_{p-3}(a) \pmod{p} \quad (1.16)$$

and

$$(2a-1) \sum_{k=0}^{p-1} \binom{-a}{k} \binom{a-1}{k} \frac{H_k^{(2)}}{2k+1} \equiv B_{p-2}(a) \pmod{p}. \quad (1.17)$$

Remark 1.2. Let $p > 3$ be a prime. As $H_{(p-1)/2}^{(2)} \equiv 0 \pmod{p}$, the number $H_k^{(2)}/(2k+1)$ is a p -adic integer for every $k = 0, 1, \dots, p-1$. For any p -adic

integer a , Z.-H. Sun [S2] determined $\sum_{k=0}^{p-1} \binom{-a}{k} \binom{a-1}{k}$ and $\sum_{k=0}^{p-1} \binom{-a}{k} \binom{a-1}{k} \frac{1}{2k+1}$ (with $a \not\equiv 1/2 \pmod{p}$) modulo p^3 . Combining this with Theorem 1.2(ii), we determine

$$\sum_{k=0}^{p-1} (-1)^k \binom{p-1}{k} \binom{p+k}{k} \binom{-a}{k} \binom{a-1}{k}$$

and

$$\sum_{k=0}^{p-1} \frac{(-1)^k}{2k+1} \binom{p-1}{k} \binom{p+k}{k} \binom{-a}{k} \binom{a-1}{k}$$

modulo p^3 since

$$(-1)^k \binom{p-1}{k} \binom{p+k}{k} = \prod_{0 < j \leq k} \left(1 - \frac{p^2}{j^2}\right) \equiv 1 - p^2 H_k^{(2)} \pmod{p^4} \text{ for } k \in \mathbb{N}.$$

Corollary 1.2. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} H_k^{(2)} &\equiv -E_{p^2-p-2} \left(\frac{1}{4}\right) \pmod{p^2} \\ &\equiv -E_{p-3} \left(\frac{1}{4}\right) \pmod{p}. \end{aligned} \tag{1.18}$$

and

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} H_k^{(2)} &\equiv \frac{1}{4} \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{(2k+1)64^k} H_k^{(2)} \equiv \frac{1}{5} \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} H_k^{(2)} \\ &\equiv -4E_{p^2-p-2} \pmod{p^2} \\ &\equiv -4E_{p-3} \pmod{p}. \end{aligned} \tag{1.19}$$

We also have

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} H_k^{(2)} &\equiv \frac{1}{2} \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{(2k+1)27^k} H_k^{(2)} \equiv \frac{1}{5} \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{(2k+1)432^k} H_k^{(2)} \pmod{p^2} \\ &\equiv (3p-3)B_{p^2-p-1} \left(\frac{1}{3}\right) \pmod{p^2} \\ &\equiv -\frac{3}{2}B_{p-2} \left(\frac{1}{3}\right) \pmod{p}. \end{aligned} \tag{1.20}$$

Remark 1.3. (i) In view of (1.20) and some conjectures in [Su11], it is interesting to investigate those primes $p > 3$ with $B_{p-2}(1/3) \equiv 0 \pmod{p}$. Though we

have not found such a prime, based on heuristic arguments we conjecture that there are infinitely many such primes. Similarly, we also conjecture that there are infinitely many odd primes p with $E_{p-3}(1/4) \equiv 0 \pmod{p}$; the first such a prime is 1019.

(ii) The author [S11b, 5.12(iii)] conjectured that for any prime $p > 3$ we have

$$2 \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} - \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{(2k+1)27^k} \equiv \left(\frac{p}{3}\right) \pmod{p^4}. \quad (1.21)$$

To conclude this section, we pose two conjectures for further research.

Conjecture 1.1. *For any prime $p > 3$, we have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k+1}}{48^k} \equiv \frac{5}{12} p^2 B_{p-2} \left(\frac{1}{3}\right) \pmod{p^3}. \quad (1.22)$$

Remark 1.4. Conjecture 1.1 refines an earlier conjecture of the author [S11b, 5.14(i)].

Conjecture 1.2. *Let $p > 3$ be a prime. Then we have*

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^2}{(2k+1)16^k} H_k^{(2)} \equiv -7B_{p-3} \pmod{p}, \quad (1.23)$$

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{k16^k} H_k^{(2)} \equiv -12 \frac{H_{p-1}}{p^2} + \frac{7}{10} p^2 B_{p-5} \pmod{p^3}, \quad (1.24)$$

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} H_k^{(2)} \equiv \frac{2}{3} \cdot \frac{H_{p-1}}{p^2} + \frac{76}{135} p^2 B_{p-5} \pmod{p^3}. \quad (1.25)$$

Remark 1.5. It is known that $H_{p-1}/p^2 \equiv -B_{p-3}/3 \pmod{p}$ for each prime $p > 3$. In contrast with (1.25), the author and R. Tauraso [ST] showed that

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} \equiv \frac{8}{9} p^2 B_{p-3} \pmod{p^3} \quad \text{for any prime } p > 3.$$

We are going to show Theorem 1.1 and Corollary 1.1 in the next section. Section 3 is devoted to the proofs of Theorem 1.2 and Corollary 1.2.

2. PROOFS OF THEOREM 1.1 AND COROLLARY 1.1

Lemma 2.1. *For any positive integer k , we have the polynomial identity*

$$\binom{-x}{k} \binom{x-1}{k} + \binom{x}{k} \binom{-x-1}{k} = 2 \left(\binom{x-1}{k} \binom{-x-1}{k} - \binom{x-1}{k-1} \binom{-x-1}{k-1} \right). \quad (2.1)$$

Proof. We may deduce (2.1) in view of [S1, p.310], but here we give a direct proof.

$$\begin{aligned} & \binom{-x}{k} \binom{x-1}{k} + \binom{x}{k} \binom{-x-1}{k} \\ &= \frac{(-1)^k}{k!k!} ((x-k) \cdots (x+k-1) + (x-k+1) \cdots (x+k)) \\ &= \frac{(-1)^k}{k!k!} (x-k+1) \cdots (x+k-1) (x-k+x+k) \\ &= 2 \frac{(-1)^k}{k!k!} \cdot \frac{(x-k+1) \cdots (x+k-1)}{x} ((x-k)(x+k) + k^2) \\ &= 2 \frac{(-1)^k}{k!k!} \left(\frac{(x-k) \cdots (x+k)}{x} + k^2 \frac{(x-k+1) \cdots (x+k-1)}{x} \right) \\ &= 2 \left(\binom{x-1}{k} \binom{-x-1}{k} - \binom{x-1}{k-1} \binom{-x-1}{k-1} \right). \end{aligned}$$

This completes the proof. \square

Lemma 2.2. *For any positive integer n , we have*

$$\frac{1}{n} \sum_{k=1}^n (k^2 - kx^2) \binom{x}{k} \binom{-x}{k} = (n^2 - x^2) \binom{x}{n} \binom{-x}{n}. \quad (2.2)$$

Proof. It is easy to verify (2.2) for $n = 1$.

Now assume that (2.2) holds for a fixed positive integer n . Then

$$\begin{aligned} & \sum_{k=1}^{n+1} (k^2 - kx^2) \binom{x}{k} \binom{-x}{k} \\ &= n(n^2 - x^2) \binom{x}{n} \binom{-x}{n} + ((n+1)^2 - (n+1)x^2) \binom{x}{n+1} \binom{-x}{n+1} \\ &= (n(n+1)^2 + (n+1)^2 - (n+1)x^2) \binom{x}{n+1} \binom{-x}{n+1} \\ &= (n+1) ((n+1)^2 - x^2) \binom{x}{n+1} \binom{-x}{n+1}. \end{aligned}$$

This concludes the induction proof. \square

Proof of Theorem 1.1. Define

$$P_n(x) := \sum_{k=0}^n \binom{-x}{k} \binom{x-1}{k} H_k \quad \text{for } n = 0, 1, 2, \dots$$

With the help of (3.1), we have

$$\begin{aligned} & P_n(x) + P_n(x+1) \\ &= \sum_{k=1}^n \left(\binom{-x}{k} \binom{x-1}{k} + \binom{x}{k} \binom{-x-1}{k} \right) H_k \\ &= 2 \sum_{k=1}^n \left(\binom{x-1}{k} \binom{-x-1}{k} H_k - \binom{x-1}{k-1} \binom{-x-1}{k-1} \left(H_{k-1} + \frac{1}{k} \right) \right) \\ &= 2 \binom{x-1}{n} \binom{-x-1}{n} H_n + \frac{2}{x^2} \sum_{k=1}^n k \binom{x}{k} \binom{-x}{k}. \end{aligned}$$

Recall that $H_{p-1} \equiv 0 \pmod{p^2}$. Thus, for any p -adic integer $x \not\equiv 0 \pmod{p}$, we have

$$P_n(x) + P_n(x+1) \equiv \frac{2}{x^2} \sum_{k=1}^{p-1} k \binom{x}{k} \binom{-x}{k} \pmod{p^2}. \quad (2.3)$$

If $x \equiv 0 \pmod{p}$, then

$$\begin{aligned} P_{p-1}(x) &= \sum_{k=1}^{p-1} \frac{-x}{k} \binom{-x-1}{k-1} \binom{x-1}{k} H_k \\ &= -x \sum_{k=1}^{p-1} \frac{1}{k} \binom{-1}{k-1} \binom{-1}{k} H_k = x \sum_{k=1}^{p-1} \left(\frac{1}{k^2} + \frac{H_{k-1}}{k} \right) \\ &\equiv x \sum_{0 < j < k < p} \frac{1}{jk} = \frac{x}{2} (H_{p-1}^2 - H_{p-1}^{(2)}) \equiv 0 \pmod{p^2} \end{aligned}$$

and also $P_{p-1}(x+1) = P_{p-1}(-x) \equiv 0 \pmod{p^2}$.

In light of (2.2), for any positive integer n we have

$$\begin{aligned} & x^2 \sum_{k=1}^n k \binom{x}{k} \binom{-x}{k} + n(n^2 - x^2) \binom{x}{n} \binom{-x}{n} \\ &= \sum_{k=1}^n k^2 \binom{x}{k} \binom{-x}{k} = -x^2 \sum_{k=1}^n \binom{x-1}{k-1} \binom{-x-1}{k-1} \\ &= x^2 \binom{x-1}{n} \binom{-x-1}{n} - x^2 \sum_{k=0}^n \binom{x-1}{k} \binom{-x-1}{k} \\ &= -(n+1)^2 \binom{x}{n+1} \binom{-x}{n+1} - x^2 \sum_{k=0}^n \binom{x-1}{k} \binom{-x-1}{k}. \end{aligned}$$

Let x be any p -adic integer with $x \not\equiv 0 \pmod{p}$. Clearly

$$\binom{x}{p-1} \binom{-x}{p-1} = \frac{\prod_{r=0}^{p-2} (r^2 - x^2)}{((p-1)!)^2} \equiv 0 \pmod{p}$$

and hence

$$\begin{aligned} & ((p-1)^2 - x^2) \binom{x}{p-1} \binom{-x}{p-1} \\ & \equiv (1 - x^2) \binom{x}{p-1} \binom{-x}{p-1} = p^2 \binom{x+1}{p} \binom{1-x}{p} \equiv 0 \pmod{p^2}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{k=1}^{p-1} k \binom{x}{k} \binom{-x}{k} & \equiv - \sum_{k=0}^{p-1} \binom{x-1}{k} \binom{-x-1}{k} \\ & \equiv - \binom{p-2}{\langle x-1 \rangle_p} (1 + (-2 - (p-2))H_{p-2}) \\ & \quad + \binom{p-2}{\langle x-1 \rangle_p} (x-1 - \langle x-1 \rangle_p) H_{\langle x-1 \rangle_p} \\ & \quad + \binom{p-2}{\langle x-1 \rangle_p} (-x-1 - \langle -x-1 \rangle_p) H_{\langle -x-1 \rangle_p} \pmod{p^2}, \end{aligned}$$

with the help of [S1, Theorem 4.1]. Note that

$$1 - pH_{p-2} = 1 + \frac{p}{p-1} - pH_{p-1} \equiv 1 - p \pmod{p^2},$$

$$H_{\langle -x-1 \rangle_p} = H_{p-1-\langle x \rangle_p} = H_{p-1} - \sum_{k=1}^{\langle x \rangle_p} \frac{1}{p-k} \equiv H_{\langle x \rangle_p} \pmod{p}$$

and

$$\begin{aligned} (p-1) \binom{p-2}{\langle x-1 \rangle_p} & = \langle x \rangle_p \binom{p-1}{\langle x \rangle_p} = (-1)^{\langle x \rangle_p} \langle x \rangle_p \prod_{k=1}^{\langle x \rangle_p} \left(1 - \frac{p}{k}\right) \\ & \equiv (-1)^{\langle x \rangle_p} \langle x \rangle_p (1 - pH_{\langle x \rangle_p}) \pmod{p^2} \\ & \equiv (-1)^{\langle x \rangle_p} x \pmod{p}. \end{aligned}$$

Combining this with (2.3), we get

$$\begin{aligned} & \frac{x^2}{2} (P_{p-1}(x) + P_{p-1}(x+1)) \\ & \equiv \sum_{k=1}^{p-1} k \binom{x}{k} \binom{-x}{k} \\ & \equiv (-1)^{\langle x \rangle_p} \langle x \rangle_p (1 - pH_{\langle x \rangle_p}) - x(-1)^{\langle x \rangle_p} (x - \langle x \rangle_p) H_{\langle x \rangle_p-1} \\ & \quad + x(-1)^{\langle x \rangle_p} (x+1 + p-1 - \langle x \rangle_p) H_{\langle x \rangle_p} \\ & \equiv (-1)^{\langle x \rangle_p} x \pmod{p^2}. \end{aligned}$$

By the above, for any p -adic integer x , we have

$$P_{p-1}(x) + P_{p-1}(x+1) \equiv \begin{cases} (-1)^{\langle x \rangle_p} 2/x \pmod{p^2} & \text{if } x \not\equiv 0 \pmod{p}, \\ 0 \pmod{p^2} & \text{otherwise.} \end{cases} \quad (2.4)$$

Therefore

$$\begin{aligned} -P_{p-1}(a) &\equiv (-1)^{\langle a \rangle_p} P_{p-1}(a - \langle a \rangle_p) - P_{p-1}(a) \\ &= \sum_{0 < k \leq \langle a \rangle_p} ((-1)^k P_{p-1}(a - k) - (-1)^{k-1} P_{p-1}(a - k + 1)) \\ &\equiv \sum_{0 < k < \langle a \rangle_p} (-1)^k (-1)^{\langle a-k \rangle_p} \frac{2}{a-k} = 2(-1)^{\langle a \rangle_p} \sum_{0 < k < \langle a \rangle_p} \frac{1}{a-k} \\ &= 2(-1)^{\langle a \rangle_p} \sum_{0 < k < \langle a \rangle_p} \left(\frac{1}{\langle a \rangle_p - k} + \frac{\langle a \rangle_p - k - (a - k)}{(a - k)(\langle a \rangle_p - k)} \right) \\ &\equiv 2(-1)^{\langle a \rangle_p} \left(H_{\langle a-1 \rangle_p} + (\langle a \rangle_p - a) H_{\langle a-1 \rangle_p}^{(2)} \right) \pmod{p^2} \end{aligned}$$

and hence

$$\begin{aligned} P_{p-1}(a) &\equiv 2(-1)^{\langle a \rangle_p - 1} \sum_{0 \leq k < \langle a \rangle_p} k^{p-2} \\ &= \frac{2(-1)^{\langle a \rangle_p - 1}}{p-1} \sum_{0 \leq k < \langle a \rangle_p} (B_{p-1}(k+1) - B_{p-1}(k)) \\ &\equiv 2(-1)^{\langle a \rangle_p} (B_{p-1}(\langle a \rangle_p) - B_{p-1}) \\ &\equiv 2(-1)^{\langle a \rangle_p} (B_{p-1}(a) - B_{p-1}) \pmod{p}. \end{aligned}$$

Note also that

$$\begin{aligned} H_{\langle a-1 \rangle_p}^{(2)} &\equiv \sum_{0 \leq k < \langle a \rangle_p} k^{p-3} = \sum_{0 \leq k < \langle a \rangle_p} \frac{B_{p-2}(k+1) - B_{p-2}(k)}{p-2} \\ &= \frac{B_{p-2}(\langle a \rangle_p) - B_{p-2}}{p-2} \equiv -\frac{1}{2} B_{p-2}(a) \pmod{p}. \end{aligned}$$

So we have the desired (1.8). \square

Lemma 2.3. *Let $p > 3$ be a prime. Then*

$$\sum_{k=1}^{\lfloor p/2 \rfloor} \frac{1}{p-2k} \equiv q_p(2) - \frac{p}{2} q_p(2)^2 \pmod{p^2}, \quad (2.5)$$

$$\sum_{k=1}^{\lfloor p/3 \rfloor} \frac{1}{p-3k} \equiv \frac{q_p(3)}{2} - \frac{p}{4} q_p(3)^2 \pmod{p^2}, \quad (2.6)$$

$$\sum_{k=1}^{\lfloor p/4 \rfloor} \frac{1}{p-4k} \equiv \frac{3}{4} q_p(2) - \frac{3}{8} p q_p(2)^2 \pmod{p^2}. \quad (2.7)$$

If $p > 5$, then

$$\sum_{k=1}^{\lfloor p/6 \rfloor} \frac{1}{p-6k} \equiv \frac{q_p(3)}{4} + \frac{q_p(2)}{3} - p \left(\frac{q_p(3)^2}{8} + \frac{q_p(2)^2}{6} \right) \pmod{p^2}. \quad (2.8)$$

Proof of Corollary 1.1. It is well known that for any $k \in \mathbb{N}$ we have

$$\begin{aligned} \binom{-1/2}{k} \binom{-1/2}{k} &= \left(\frac{\binom{2k}{k}}{(-4)^k} \right)^2 = \frac{\binom{2k}{k}^2}{16^k}, \\ \binom{-1/3}{k} \binom{-2/3}{k} &= \frac{\binom{2k}{k} \binom{3k}{k}}{27^k}, \\ \binom{-1/4}{k} \binom{-3/4}{k} &= \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k}, \\ \binom{-1/6}{k} \binom{-5/6}{k} &= \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k}. \end{aligned}$$

Applying the first congruence in (1.8) with $a = 1/2$ and Lehmer's congruence (2.5), we obtain

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} H_k &\equiv (-1)^{(p-1)/2} 2 \sum_{k=1}^{(p-1)/2} \frac{1}{1/2 - k} \\ &= -2 \left(\frac{-1}{p} \right) \sum_{\substack{j=1 \\ 2 \nmid j}}^{p-1} \frac{1}{j} = -2 \left(\frac{-1}{p} \right) \sum_{k=1}^{\lfloor p/2 \rfloor} \frac{1}{p-2k} \\ &\equiv -2 \left(\frac{-1}{p} \right) \left(q_p(2) - \frac{p}{2} q_p(2)^2 \right) \pmod{p^2}. \end{aligned}$$

This proves (1.10). Choose $r \in \{1, 2\}$ with $r \equiv -p \pmod{3}$. Then $\langle r/3 \rangle_p = (p+r)/3$. By the first congruence in (1.8) with $a = r/3$ and Lehmer's congruence (2.6), we have

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} H_k &\equiv (-1)^{(p+r)/3-1} 2 \sum_{0 < k < (p+r)/3} \frac{1}{r/3 - k} \\ &= -6(-1)^r \sum_{\substack{j=1 \\ 3 \nmid j+r}}^{p-1} \frac{1}{j} = -6 \left(\frac{p}{3} \right) \sum_{k=1}^{\lfloor p/3 \rfloor} \frac{1}{p-3k} \\ &\equiv -6 \left(\frac{p}{3} \right) \left(\frac{q_p(3)}{2} - \frac{p}{4} q_p(3)^2 \right) \pmod{p^2}. \end{aligned}$$

This proves (1.11). Choose $s \in \{1, 3\}$ with $s \equiv -p \pmod{4}$. Then $\langle s/4 \rangle_p = (p+s)/4$. By the first congruence in (1.8) with $a = s/4$ and Lehmer's congruence (2.7), we have

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} H_k &\equiv (-1)^{(p+s)/4-1} 2 \sum_{0 < k < (p+s)/4} \frac{1}{s/4 - k} \\ &= 8(-1)^{(p+s)/4} \sum_{\substack{j=1 \\ 4|j+s}}^{p-1} \frac{1}{j} = -8 \left(\frac{-2}{p} \right) \sum_{k=1}^{\lfloor p/4 \rfloor} \frac{1}{p-4k} \\ &\equiv -8 \left(\frac{-2}{p} \right) \left(\frac{3}{4} q_p(2) - \frac{3}{8} p q_p(2)^2 \right) \pmod{p^2}. \end{aligned}$$

This proves (1.12). Choose $t \in \{1, 5\}$ with $t \equiv -p \pmod{6}$. Then $\langle t/6 \rangle_p = (p+t)/6$. Provided $p > 5$, by the first congruence in (1.8) with $a = t/6$ and Lehmer's congruence (2.8), we have

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} H_k &\equiv (-1)^{(p+t)/6-1} 2 \sum_{0 < k < (p+t)/6} \frac{1}{t/6 - k} \\ &= 12(-1)^{(p+t)/6} \sum_{\substack{j=1 \\ 6|j+t}}^{p-1} \frac{1}{j} = -12 \left(\frac{-1}{p} \right) \sum_{k=1}^{\lfloor p/6 \rfloor} \frac{1}{p-6k} \\ &\equiv -12 \left(\frac{-1}{p} \right) \left(\frac{q_p(3)}{4} + \frac{q_p(2)}{3} - p \left(\frac{q_p(3)^2}{8} + \frac{q_p(2)^2}{6} \right) \right) \pmod{p^2}. \end{aligned}$$

This proves (1.13). (Note that (1.13) for $p = 5$ can be verified directly.) We are done. \square

3. PROOFS OF THEOREM 1.2 AND COROLLARY 1.2

For any $n \in \mathbb{N}$, we define

$$W_n(x) := \sum_{k=0}^n \binom{-x}{k} \binom{x-1}{k} H_k^{(2)} \quad \text{and} \quad w_n(x) := \sum_{k=0}^n \binom{-x}{k} \binom{x-1}{k} \frac{H_k^{(2)}}{2k+1}. \quad (3.1)$$

Lemma 3.1. *For any $n \in \mathbb{N}$ we have*

$$W_n(x) + W_n(x+1) = 2 \binom{x-1}{n} \binom{-x-1}{n} \left(H_n^{(2)} + \frac{1}{x^2} \right) - \frac{2}{x^2} \quad (3.2)$$

and

$$(2x+1)w_n(x+1) - (2x-1)w_n(x) = 2 \binom{x-1}{n} \binom{-x-1}{n} \left(H_n^{(2)} + \frac{1}{x^2} \right) - \frac{2}{x^2}. \quad (3.3)$$

Proof. For any positive integer k , there is the polynomial identity

$$\begin{aligned} & (2x+1) \binom{x}{k} \binom{-x-1}{k} - (2x-1) \binom{-x}{k} \binom{x-1}{k} \\ &= 2(2k+1) \left(\binom{x-1}{k} \binom{-x-1}{k} - \binom{x-1}{k-1} \binom{-x-1}{k-1} \right). \end{aligned} \quad (3.4)$$

In fact,

$$\begin{aligned} & (2x+1) \binom{x}{k} \binom{-x-1}{k} - (2x-1) \binom{-x}{k} \binom{x-1}{k} \\ &= \frac{(-1)^k}{k!k!} ((2x+1)(x-k+1) \cdots (x+k) - (2x-1)(x-k) \cdots (x+k-1)) \\ &= \frac{(-1)^k}{k!k!} (x-k+1) \cdots (x+k-1) ((2x+1)(x+k) - (2x-1)(x-k)) \\ &= (-1)^k \frac{2(2k+1)}{k!k!} \cdot \frac{(x-k+1) \cdots (x+k-1)}{x} ((x-k)(x+k) + k^2) \\ &= 2(2k+1) \left(\binom{x-1}{k} \binom{-x-1}{k} - \binom{x-1}{k-1} \binom{-x-1}{k-1} \right). \end{aligned}$$

In light of (2.1) and (3.4),

$$\begin{aligned} & W_n(x) + W_n(x+1) \\ &= (2x+1)w_n(x) - (2x-1)w_n(x+1) \\ &= 2 \sum_{k=1}^n \left(\binom{x-1}{k} \binom{-x-1}{k} H_k^{(2)} - \binom{x-1}{k-1} \binom{-x-1}{k-1} \left(H_{k-1}^{(2)} + \frac{1}{k^2} \right) \right) \\ &= 2 \binom{x-1}{n} \binom{-x-1}{n} H_n^{(2)} - 2 \sum_{k=1}^n \frac{1}{k^2} \binom{x-1}{k-1} \binom{-x-1}{k-1} \\ &= 2 \binom{x-1}{n} \binom{-x-1}{n} H_n^{(2)} + \frac{2}{x^2} \sum_{k=1}^n \binom{x}{k} \binom{-x}{k}. \end{aligned}$$

Combining this with the identity

$$\sum_{k=0}^n \binom{x}{k} \binom{-x}{k} = \binom{x-1}{n} \binom{-x-1}{n} \quad (3.5)$$

we immediately obtain (3.2) and (3.3). Note that the polynomial identity (3.5) holds if and only if it is valid for all $x = -n, -n-1, \dots$. For each $x = -n, -n-1, \dots$, the identity (3.5) has the equivalent form

$$\sum_{k=0}^n \binom{x}{k} \binom{-x}{-x-k} = \binom{x-1}{n} \binom{-x-1}{n}$$

which is a special case of Andersen's identity

$$m \sum_{k=0}^n \binom{x}{k} \binom{-x}{m-k} = (m-n) \binom{x-1}{n} \binom{-x}{m-n} \quad (m \geq n \geq 0) \quad (3.6)$$

(cf. (3.14) of [G, p. 23]). This concludes the proof. \square

Lemma 3.2. *Let p be any prime, and let x be a nonzero p -adic integer. Then*

$$\binom{x-1}{p-1} \binom{-x-1}{p-1} \left(H_{p-1}^{(2)} + \frac{1}{x^2} \right) - \frac{1}{x^2} \equiv \begin{cases} -1/x^2 \pmod{p^2} & \text{if } x \not\equiv 0 \pmod{p}, \\ 0 \pmod{p^2} & \text{otherwise.} \end{cases} \quad (3.7)$$

Proof. If $x \not\equiv 0 \pmod{p}$, then

$$\binom{x-1}{p-1} \binom{-x-1}{p-1} = \frac{p^2}{-x^2} \binom{x}{p} \binom{-x}{p} \equiv 0 \pmod{p^2}$$

and hence (3.7) holds.

Below we assume $x \equiv 0 \pmod{p}$. Write $x = p^n x_0$, where n is a positive integer and x_0 is a p -adic integer with $x_0 \not\equiv 0 \pmod{p}$. Clearly,

$$\begin{aligned} \binom{x-1}{p-1} \binom{-x-1}{p-1} &= \prod_{k=1}^{p-1} \left(\frac{p^n x_0 - k}{k} \cdot \frac{-p^n x_0 - k}{k} \right) = \prod_{k=1}^{p-1} \left(1 - \frac{p^{2n} x_0^2}{k^2} \right) \\ &\equiv 1 - \sum_{k=1}^{p-1} \frac{p^{2n} x_0^2}{k^2} = 1 - x^2 H_{p-1}^{(2)} \pmod{p^{4n}}. \end{aligned}$$

and hence

$$\frac{\binom{x-1}{p-1} \binom{-x-1}{p-1} - 1}{x^2} \equiv -H_{p-1}^{(2)} \pmod{p^{2n}}.$$

Therefore,

$$\binom{x-1}{p-1} \binom{-x-1}{p-1} \left(H_{p-1}^{(2)} + \frac{1}{x^2} \right) - \frac{1}{x^2} \equiv H_{p-1}^{(2)} + \frac{\binom{x-1}{p-1} \binom{-x-1}{p-1} - 1}{x^2} \equiv 0 \pmod{p^2}.$$

This completes the proof. \square

Proof of Theorem 1.2. (i) Let m be any positive integer with $a \equiv m \pmod{p^2}$. Obviously,

$$W_{p-1}(a) \equiv W_{p-1}(m) \pmod{p^2} \quad \text{and} \quad w_{p-1}(a) \equiv w_{p-1}(m) \pmod{p^2}.$$

In light of Lemmas 3.1 and 3.2,

$$\begin{aligned}
& (-1)^m W_{p-1}(m) - (-1)^1 W_{p-1}(1) \\
&= \sum_{0 < k < m} ((-1)^{k+1} W_{p-1}(k+1) - (-1)^k W_{p-1}(k)) \\
&\equiv \sum_{\substack{0 < k < m \\ p \nmid k}} (-1)^{k+1} \frac{-2}{k^2} \pmod{p^2}
\end{aligned}$$

and

$$\begin{aligned}
& (2m-1)W_{p-1}(m) - (2 \times 1 - 1)W_{p-1}(1) \\
&= \sum_{0 < k < m} ((2k+1)W_{p-1}(k+1) - (2k-1)W_{p-1}(k)) \\
&\equiv \sum_{\substack{0 < k < m \\ p \nmid k}} \frac{-2}{k^2} \pmod{p^2}.
\end{aligned}$$

Note that $W_{p-1}(1) = 0 = w_{p-1}(1)$. So we have the first congruence in (1.14) as well as the first congruence in (1.15).

It is well-known that $E_n(x) + E_n(x+1) = 2x^n$ and $E_{2n+2}(0) = 0$ for all $n \in \mathbb{N}$. Let φ be Euler's totient function. Then

$$\begin{aligned}
2 \sum_{0 < k < m/p \nmid k} \frac{(-1)^{m-k}}{k^2} &\equiv 2 \sum_{k=0}^{m-1} (-1)^{m-k} k^{\varphi(p^2)-2} \\
&= (-1)^m \sum_{k=0}^{m-1} (-1)^k (E_{\varphi(p^2)-2}(k) + E_{\varphi(p^2)-2}(k+1)) \\
&= (-1)^m \sum_{k=0}^{m-1} ((-1)^k E_{\varphi(p^2)-2}(k) - (-1)^{k+1} E_{\varphi(p^2)-2}(k+1)) \\
&= (-1)^m (E_{\varphi(p^2)-2}(0) - (-1)^m E_{\varphi(p^2)-2}(m)) = -E_{\varphi(p^2)-2}(m) \\
&\equiv -E_{\varphi(p^2)-2}(a) \pmod{p^2}.
\end{aligned}$$

This proves the second congruence in (1.14).

To complete the proof of (1.15), we only need to show that if m is a positive integer with $a \equiv m \pmod{p^3}$ then

$$\sum_{\substack{0 < k < m \\ p \nmid k}} \frac{1}{k^2} \equiv (p-1)B_{\varphi(p^2)-1}(a) \pmod{p^2}.$$

In fact, as $B_n(x+1) - B_n(x) = nx^{n-1}$ and $B_{2n+3} = 0$ for all $n \in \mathbb{N}$, and pB_n is a p -adic integer for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{\substack{0 < k < m \\ p \nmid k}} \frac{1}{k^2} &\equiv \sum_{k=0}^{m-1} k^{\varphi(p^2)-2} = \sum_{k=0}^{m-1} \frac{B_{\varphi(p^2)-1}(k+1) - B_{\varphi(p^2)-1}(k)}{\varphi(p^2) - 1} \\
&= \frac{B_{\varphi(p^2)-1}(m) - B_{\varphi(p^2)-1}}{\varphi(p^2) - 1} = \frac{B_{\varphi(p^2)-1}(m)}{p^2 - p - 1} \\
&\equiv (p-1)B_{\varphi(p^2)-1}(a) + (p-1) \sum_{k=0}^{\varphi(p^2)-1} \binom{\varphi(p^2)-1}{k} (pB_{\varphi(p^2)-1-k}) \frac{m^k - a^k}{p} \\
&\equiv (p-1)B_{\varphi(p^2)-1}(a) \pmod{p^2}.
\end{aligned}$$

(ii) Choose $m \in \{1, 2, \dots, p^2\}$ such that $a \equiv m \pmod{p^2}$. Write $m = ps + r$ with $s \in \{0, \dots, p-1\}$ and $r \in \{1, \dots, p\}$. Then, for any $\varepsilon = \pm 1$ we have

$$\begin{aligned}
\sum_{\substack{0 < k < m \\ p \nmid k}} \frac{\varepsilon^k}{k^2} &= \sum_{k=1}^s \sum_{t=1}^{p-1} \frac{\varepsilon^{pk-t}}{(pk-t)^2} + \sum_{0 < t < r} \frac{\varepsilon^{ps+t}}{(ps+t)^2} \\
&\equiv \sum_{k=1}^s \varepsilon^k \sum_{t=1}^{p-1} \frac{\varepsilon^t}{t^2} + \varepsilon^s \sum_{0 < t < r} \frac{\varepsilon^t}{t^2} \\
&\equiv \varepsilon^s \sum_{t=0}^{r-1} \varepsilon^t t^{p-3} \pmod{p}
\end{aligned}$$

since $H_{p-1}^{(2)} \equiv 0 \pmod{p}$ and

$$\sum_{t=1}^{p-1} \frac{(-1)^t}{t^2} = \sum_{t=1}^{(p-1)/2} \left(\frac{(-1)^t}{t^2} + \frac{(-1)^{p-t}}{(p-t)^2} \right) \equiv 0 \pmod{p}.$$

Thus, we deduce from (1.14) that

$$\begin{aligned}
W_{p-1}(a) &\equiv 2(-1)^m \sum_{\substack{0 < k < m \\ p \nmid k}} \frac{(-1)^k}{k^2} \equiv (-1)^r \sum_{t=0}^{r-1} (-1)^t (2t^{p-3}) \\
&= (-1)^r \sum_{t=0}^{p-1} ((-1)^t E_{p-3}(t) - (-1)^{t+1} E_{p-3}(t+1)) \\
&= (-1)^r (E_{p-3}(0) - (-1)^r E_{p-3}) = -E_{p-3}(r) \\
&\equiv -E_{p-3}(a) \pmod{p}.
\end{aligned}$$

Similarly, from (1.15) we obtain

$$\begin{aligned}
(2a-1)w_{p-1}(a) &\equiv -2 \sum_{\substack{0 < k < m \\ p \nmid k}} \frac{1}{k^2} \equiv \sum_{t=0}^{r-1} ((p-2)t^{p-3}) \\
&= \sum_{t=0}^{p-1} (B_{p-2}(t+1) - B_{p-2}(t)) = B_{p-2}(r) \\
&\equiv B_{p-2}(a) \pmod{p}.
\end{aligned}$$

Therefore both (1.16) and (1.17) are valid.

By the above, we have completed the proof of Theorem 1.2. \square

Lemma 3.3. *For any prime $p > 3$, we have*

$$\sum_{k=1}^{(p^2-1)/2} \frac{1}{k^2} \equiv \sum_{k=1}^{p^2-1} \frac{1}{k^2} \equiv 0 \pmod{p^2}. \quad (3.8)$$

Proof. Since $\{2j : 0 < j < p^2 \text{ \& } p \nmid j\}$ is a reduced system of residues modulo p^2 , we have

$$\sum_{\substack{j=1 \\ p \nmid j}}^{p^2-1} \frac{1}{(2j)^2} \equiv \sum_{\substack{k=1 \\ p \nmid k}}^{p^2-1} \frac{1}{k^2} \pmod{p^2} \quad \text{and hence} \quad \sum_{\substack{k=1 \\ p \nmid k}}^{p^2-1} \frac{1}{k^2} \equiv 0 \pmod{p^2}.$$

Note also that

$$2 \sum_{\substack{k=1 \\ p \nmid k}}^{(p^2-1)/2} \frac{1}{k^2} \equiv \sum_{\substack{k=1 \\ p \nmid k}}^{(p^2-1)/2} \left(\frac{1}{k^2} + \frac{1}{(p^2-k)^2} \right) = \sum_{\substack{k=1 \\ p \nmid k}}^{p^2-1} \frac{1}{k^2} \pmod{p^2}.$$

Therefore (3.8) holds. \square

Proof of Corollary 1.2. Applying (1.14) and (1.16) with $a = 1/4$ we immediately get (1.18).

For every $a = 1, 2, \dots$, we obviously have

$$E_{\varphi(p^a)-2} \left(\frac{1}{2} \right) = \frac{E_{\varphi(p^a)-2}}{2^{\varphi(p^a)-2}} \equiv 4E_{\varphi(p^a)-2} \pmod{p^a}.$$

(1.14) and (1.16) with $a = 1/2$ yield

$$\begin{aligned}
\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} H_k^{(2)} &\equiv -4E_{p^2-p-2} \pmod{p^2} \\
&\equiv -4E_{p-3} \pmod{p}.
\end{aligned}$$

Clearly $1/2 \equiv (p^2 + 1)/2 \pmod{p^2}$ and $1/4 \equiv (3p^2 + 1)/4 \pmod{p^2}$. Applying Theorem 1.2(i) with $a = 1/2, 1/4$, we get

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} H_k^{(2)} \equiv -2 \sum_{\substack{k=1 \\ p \nmid k}}^{(p^2-1)/2} \frac{(-1)^k}{k^2} \pmod{p^2} \quad (3.9)$$

and

$$-\frac{1}{2} \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{(2k+1)64^k} H_k^{(2)} \equiv -2 \sum_{\substack{k=1 \\ p \nmid k}}^{3(p^2-1)/4} \frac{1}{k^2} \pmod{p^2}. \quad (3.10)$$

In view of (3.8),

$$\begin{aligned} \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{3}{4}(p^2-1)} \frac{1}{k^2} &= \sum_{\substack{k=1 \\ p \nmid k}}^{p^2-1} \frac{1}{k^2} - \sum_{\substack{k=1 \\ p \nmid k}}^{(p^2-1)/4} \frac{1}{(p^2-k)^2} \\ &\equiv -2 \sum_{\substack{k=1 \\ p \nmid k}}^{(p^2-1)/4} \frac{2}{(2k)^2} = -2 \sum_{\substack{k=1 \\ p \nmid k}}^{(p^2-1)/2} \frac{1+(-1)^k}{k^2} \\ &\equiv -2 \sum_{\substack{k=1 \\ p \nmid k}}^{(p^2-1)/2} \frac{(-1)^k}{k^2} \pmod{p^2}. \end{aligned}$$

Combining this with (3.9) and (3.10) we obtain the first congruence in (1.19). It is known that

$$E_n \left(\frac{1}{6} \right) = 2^{-n-1} (1 + 3^{-n}) E_n \quad \text{for all } n = 0, 2, 4, 6, \dots$$

(see, e.g., G. J. Fox [F]). Thus, applying (1.14) with $a = 1/6$ we get

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} H_k^{(2)} &\equiv -E_{p^2-p-2} \left(\frac{1}{6} \right) = -2^{-\varphi(p^2)+1} \left(1 + 3^{-\varphi(p^2)+2} \right) E_{p^2-p-2} \\ &\equiv -20E_{p^2-p-2} \pmod{p^2} \end{aligned}$$

and hence

$$\frac{1}{5} \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} H_k^{(2)} \equiv -4E_{p^2-p-2} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} H_k^{(2)} \pmod{p^2}.$$

(The first congruence in the last formula can be verified directly for $p = 5$.) This concludes the proof of (1.19).

By (1.15) and (1.17) in the case $a = 1/3$, we have

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{(2k+1)27^k} H_k^{(2)} &\equiv -3(2-2p)B_{p^2-p-1} \left(\frac{1}{3} \right) \pmod{p^2} \\ &\equiv -3B_{p-2} \left(\frac{1}{3} \right) \pmod{p}. \end{aligned}$$

Below we show the first two congruences in (1.20) for $p > 5$. (The case $p = 5$ can be checked directly.) Clearly $1/3 \equiv (2p^2+1)/3 \pmod{p^2}$. Applying (1.14) and (1.15) with $a = 1/3$ and $m = (2p^2+1)/3$, we obtain

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} H_k^{(2)} \equiv -2 \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{2}{3}(p^2-1)} \frac{(-1)^k}{k^2} \pmod{p^2} \quad (3.11)$$

and

$$-\frac{1}{3} \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{(2k+1)27^k} H_k^{(2)} \equiv -2 \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{2}{3}(p^2-1)} \frac{1}{k^2} \pmod{p^2}. \quad (3.12)$$

On the other hand, (1.9) with $a = 1/6$ and $m = (5p^2+1)/6$ yields

$$-\frac{2}{3} \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{(2k+1)432^k} H_k^{(2)} \equiv -2 \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{5}{6}(p^2-1)} \frac{1}{k^2} \pmod{p^2}. \quad (3.13)$$

Observe that

$$\begin{aligned} 2 \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{2}{3}(p^2-1)} \frac{(-1)^k + 1}{k^2} &= 2 \sum_{\substack{j=1 \\ p \nmid j}}^{(p^2-1)/3} \frac{2}{(2j)^2} \\ &\equiv \sum_{\substack{j=1 \\ p \nmid j}}^{(p^2-1)/3} \frac{1}{(p^2-j)^2} = \sum_{\substack{k=1 \\ p \nmid k}}^{p^2-1} \frac{1}{k^2} - \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{2}{3}(p^2-1)} \frac{1}{k^2} \\ &\equiv - \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{2}{3}(p^2-1)} \frac{1}{k^2} \pmod{p^2}. \end{aligned}$$

Thus

$$2 \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{2}{3}(p^2-1)} \frac{(-1)^k}{k^2} \equiv -3 \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{2}{3}(p^2-1)} \frac{1}{k^2} \pmod{p^2} \quad (3.14)$$

and

$$\sum_{\substack{k=1 \\ p \nmid k}}^{(p^2-1)/3} \frac{1}{k^2} \equiv - \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{2}{3}(p^2-1)} \frac{1}{k^2} \equiv \frac{2}{3} \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{2}{3}(p^2-1)} \frac{(-1)^k}{k^2} \pmod{p^2}. \quad (3.15)$$

With the help of (3.8), we have

$$\begin{aligned} \sum_{\substack{k=1 \\ p \nmid k}}^{(p^2-1)/3} \frac{(-1)^k}{k^2} &= \sum_{\substack{k=1 \\ p \nmid k}}^{p^2-1} \frac{(-1)^k}{k^2} - \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{2}{3}(p^2-1)} \frac{(-1)^{p^2-k}}{(p^2-k)^2} \\ &\equiv \sum_{\substack{k=1 \\ p \nmid k}}^{p^2-1} \frac{(-1)^k + 1}{k^2} + \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{2}{3}(p^2-1)} \frac{(-1)^k}{k^2} \\ &\equiv \sum_{\substack{k=1 \\ p \nmid k}}^{(p^2-1)/2} \frac{2}{(2k)^2} + \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{2}{3}(p^2-1)} \frac{(-1)^k}{k^2} \pmod{p^2} \end{aligned}$$

and hence

$$\sum_{\substack{k=1 \\ p \nmid k}}^{(p^2-1)/3} \frac{(-1)^k}{k^2} \equiv \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{2}{3}(p^2-1)} \frac{(-1)^k}{k^2} \pmod{p^2}. \quad (3.16)$$

Adding (3.15) and (3.16) we get

$$\sum_{\substack{k=1 \\ p \nmid k}}^{(p^2-1)/6} \frac{2}{(2k)^2} \equiv \frac{5}{3} \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{2}{3}(p^2-1)} \frac{(-1)^k}{k^2} \pmod{p^2}$$

and hence

$$\begin{aligned} \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{5}{6}(p^2-1)} \frac{1}{k^2} &= \sum_{\substack{k=1 \\ p \nmid k}}^{p^2-1} \frac{1}{k^2} - \sum_{\substack{k=1 \\ p \nmid k}}^{(p^2-1)/6} \frac{1}{(p^2-k)^2} \\ &\equiv - \sum_{\substack{k=1 \\ p \nmid k}}^{(p^2-1)/6} \frac{1}{k^2} \equiv -\frac{10}{3} \sum_{\substack{k=1 \\ p \nmid k}}^{\frac{2}{3}(p^2-1)} \frac{(-1)^k}{k^2} \pmod{p^2}. \end{aligned}$$

Combining this with (3.11)-(3.14) we obtain the first two congruences in (1.20). (When $p = 5$, the second congruence in (1.20) can be verified directly.) This concludes the proof. \square

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